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ROTATIONAL INVARIANCE OF MAXWELL'S EQUATIONS.

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It is shown that Maxwell's vector equations are rotationally invariant in the usual sense; that is, they assume the same vector form in each of two orthogonal coordinate systems which experience a relative angular velocity. This conclusion depends upon the well-known invariance of the Minkowski tensor formulation of Maxwell's equations and upon the heretofore uncertain orthogonality and flatness of space-time as seen by a rotating observer. Because there has been some question in the past regarding the curvature of the rotating space, this matter is first resolved by pointing out an apparently unnoticed but, nevertheless, fundamental distinction between transformations in space only and those in space-time. Once the properties of space-time transformations are established, it then follows that the rotating space is Euclidean even though time and circumferential distance undergo Lorentz contractions. AUTHOR

Introduction. A rotational transformation to determine the form of Maxwell's equations in a spinning system was first attempted by Schiff [1][1] and later by Trocheris [2]. In this second analysis, Trocheris attempted to modify Schiff's conclusions by partially accounting for the Lorentz contraction; but by not distinguishing between transformation parameters and space variables, both he and Schiff were led to field quantities which were expressed in terms of non-orthogonal rotating coordinate systems, thus masking the invariance of Maxwell's equations. Such difficulties may be avoided by noting that the Lorentz transformation may be applied between systems k_1 and k_2 without altering the total curvature of the space as determined for the preferred, or stationary, system k_1. In other words, the curvature of the space may alter only if we wish to establish a second preferred system, say k_3.

It has previously been assumed that k_2 may undergo accelerations without violating the conditions of special relativity; that is, that any changes in length only depend upon the velocity and that any differences in elapsed time, as measured by observers in k_1 and k_2, depend upon the velocity and the velocity history, being unaffected by accelerations, except that they serve to change the velocities. This assumption, stated very early by Einstein [3], has recently been supported by experiments reported by Sherwin [4].

Relativistic rotations themselves have been studied by Thirring [5], Berenda

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1) Numbers in brackets refer to the references at the end of the paper.

[6], Hill [7], Møller [8], Rosen [9], Riabushko and Fisher [10], Hönl and Dehnen [11], and Phipps [12]. Thirring, who was concerned with the effect of rotation of distant matter upon the curvature of space, introduced a skewed transformation (defined in the next section), equation (23), so that his subsequent results reflect both the effect of matter and the effect of the skewed transformation. Berenda, on the other hand, was interested in the Lorentz contraction of a rotating disk. In the course of his analysis he employed the same transformation used by Thirring, and thus introduced an artificial curvature into the transformed space. Hill approached the circumferential contraction of a rotating disk by associating the tangential velocity at the periphery of the disk with the linear velocity of an observer in rectilinear motion, and so there was no attempt at studying the metric in the rotating frame of reference. He did conclude, however, that the speed-distance relationship for the disk must be non-linear. While Møller pointed out the change in length in the circumferential direction as a result of the motion, he neglected a term in the associated time transformation. Rosen's study initially employed the metric of the rotating space without the Lorentz contraction. These effects were then introduced in such a way as to maintain the distortion previously found by Berenda. Riabushko and Fisher, as well as Hönl and Dehnen returned to Thirring's skewed transformation in their investigation of rotation within the general theory of relativity. Phipps misinterpreted the nature of the Lorentz transformation early in his development, and consequently was led to an incorrect sign in his final expression.

Since the analyses of Schiff and Trocheris seem to be the only ones in the literature that are directly concerned with the form of Maxwell's equations, the results of the present work will be compared with theirs. Recently Webster [13] attempted to show that Schiff's results may be satisfactory for a so-called first-order explanation of certain specific problems. The present results enable these approximate solutions to be easily replaced by exact expressions.

The geometry of four-dimensional transformations. Attention in this section will be focused upon three spaces and upon two transformation types. We shall consider the Euclidean three-space E^3, in which most experiments are interpreted, as a hypersurface in a four-dimensional Euclidean space E^4, as well as in a four-dimensional space R^4. Thus the hypersurface E^3 is the intersection of E^4 and R^4, which differ in that the metric associated with E^4 is positive definite, while that associated with R^4 is not. In particular, the arc length ds may be represented as

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$$

in E^4, and as

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2(dx^4)^2$$

in R^4. In these relations the x^i are merely generic coordinates, and c is the velocity of light as measured in the coordinate system used to span R^4.

One of the transformation types will be denoted by ${}^3T$, ${}_3T$, or ${}^3_3T$; and the other will be denoted by ${}^4T$, ${}_4T$, or ${}^4_4T$; each trio corresponds to contravariant, covariant, or mixed transformations, respectively. Individual transformations may refer to translation, or rotation, or a combination thereof. For brevity, only the contravariant form, which is typical of the Lorentz transformation, will be used. Further definition of these transformations will be deferred for the moment.

It will also prove convenient to consider five coordinate systems: a stationary rectangular cartesian system x^i, a non-stationary rectangular cartesian system y^i, a rotating rectangular cartesian system z^i, a stationary cylindrical system X^i, and a rotating cylindrical system Z^i. In this statement, and in what follows, the usual notation of differential geometry will be used, as defined in various texts, such as Eisenhart **[14]**. The cylindrical and cartesian systems will be used to demonstrate the basic differences between transformation types, but, for simplicity, only rectangular cartesian systems will be used in the formulation of Maxwell's equations.

The essential differences between transformations ${}^3T$ and ${}^4T$ in E^4 and in R^4 are demonstrable in any of the coordinate systems listed above. Initially a stationary cylindrical coordinate system X^i in E^3 and a moving cylindrical coordinate system Z^i, also in E^3, whose origin is coincident with that of X^i, and whose positive Z^2 axis is coincident with the positive X^2 axis, will be considered. The rotation of the Z^i system is conventionally expressed by a relation of the form

$$(1) \qquad Z^1 = X^1, \quad Z^2 = X^2 + \omega t, \quad Z^3 = X^3,$$

where ω and t represent the angular velocity and time, in that order. The metric associated with Z^i may be determined from (1) and the metric associated with X^i, which is $G_{11}=1$, $G_{22}=(X^1)^2$, $G_{33}=1$, all other $G_{ij}=0$. Upon carrying out this calculation we, of course, find that $K_{11}=1$, $K_{22}=(Z^1)^2$, $K_{33}=1$, all other $K_{ij}=0$.

Suppose that this computation is now carried out in E^4. The higher space may be formed from E^3 by adjoining the coordinate X^4, the time variable, and setting $G_{44}=1$. The choice of $G_{44}=1$, rather than -1, is to avoid confusion with the space R^4 in which G_{44} is negative. The extension of relation (1) will parallel that used by Schiff if the relations

$$(2) \qquad Z^1 = X^1, \quad Z^2 = X^2 + \omega X^4, \quad Z^3 = X^3, \quad Z^4 = X^4$$

are imposed.

It is easy to verify that by using (2) and the metric G_{ij}, just described, that the metric K_{ij} will have components

$$(3) \qquad K_{11}=1\,,\ \ K_{22}=(Z^1)^2\,,\ \ K_{33}=1\,,\ \ K_{44}=1+\omega^2(Z^1)^2\,,\ \ K_{24}=-(Z^1)^2\,,$$

which clearly indicates that the Z^i no longer describe an orthogonal system.

Since (2) is merely a formal extension of (1), entirely divorced of any particular physical significance, it is clear that the non-orthogonality indicated by (3) must be due entirely to the form of (2)[2]. This is indeed the case. The key to the trouble is that X^4 played no part in determining K_{ij}, $i,j=1,2,3$, in E^3 because it is not in E^3. Moving from the hypersurface E^3 into the enveloping space E^4, however, demands that X^4 now play a role in determining K_{ij}. To an observer in E^4, relation (2) clearly states that the Z^i system is composed of coordinates Z^1, Z^3, and Z^4 which are equal to coordinates X^1, X^3, and X^4 of the orthogonal system X^i, and, therefore, are themselves mutually orthogonal. On the other hand, Z^2, according to (2), is a linear combination of X^2 and X^4, and hence Z^2 cannot be orthogonal to Z^4. A non-zero value of K_{24} and K_{42} is then to be expected, since it is related to the cosine, in E^4, of the angle between Z^2 and Z^4. The nature of K_{44} also reflects this non-orthogonality, as should be expected, since a motion along the Z^2 axis may be accompanied by a motion along the Z^4 axis.

A similar distortion may be experienced in the case of linear translation with a constant velocity v. Suppose, for instance, that the transformation is of the form

$$(4) \qquad \begin{gathered} x^1=y^1\,,\quad x^2=y^2\,,\quad x^3=y^3+vy^4\,,\quad x^4=y^4\,,\\ (ds)^2=(dx^1)^2+(dx^2)^2+(dx^3)^2-c^2(dx^4)^2\,; \end{gathered}$$

it is then easy to show that

$$a_{11}=a_{22}=a_{33}=1\,,\qquad a_{34}=a_{43}=v\,,\qquad a_{44}=-c^2\left(1-\frac{v^2}{c^2}\right)$$

and that all other $a_{ij}=0$, such that $(ds)^2=a_{ij}dy^i dy^j$. Certainly this distortion is not due to motion of the matter of the universe. It is not only contrary to experimental evidence, but it is due entirely to the nature of the transformation.

Once the trouble arising from (2) has been diagnosed, it can be seen that the correct form in E^4 for a transformation in E^4, yielding the relation imperfectly expressed in (1), is

$$(5) \qquad Z^1=X^1\,,\quad Z^2=X^2-\phi\,,\quad Z^3=X^3\,,\quad Z^4=X^4\,.$$

Here ϕ is viewed as a transformation parameter which may take on the value

2) The choice $G_{44}=-1$ yields $K_{44}=-1+\omega^2(Z^1)^2$ with no change in the remaining K_{ij}.

ωt, but which is not associated with the coordinates X^4 or Z^4. Upon recalling that $\omega t/2\pi = \nu$, where ν, not generally an integer, measures the extent of the rotation ($\nu = \frac{\theta}{2\pi}$ mod n, where n is the number of complete rotations, and θ is the angular position in radians from the reference direction), it is evident that since ν is not a geometrical entity in E^4, neither is ωt.

Although the major concern is with rotating coordinate systems, it is helpful at this juncture to note that the correct form in E^4 of a translational transformation corresponding to a relative motion along the X^3 axis in E^3 is

$$(6) \qquad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3 - \phi, \quad y^4 = x^4,$$

in which $\phi = vt$, such that t represents the time during which the relative velocity v is maintained.

As these examples indicate, an observer in E^4 sees coordinate transformations of the above type merely as coordinate shifts in the E^3 hypersurface. These transformations will be defined as type ${}^3T$; i. e., transformations confined to a hypersurface normal to the time axis. Transformations which result in coordinate shifts having components along the time axis will be defined as type ${}^4T$.

Based upon experience in E^3, it is possible to further classify type ${}^4T$ transformations into two classes: proper and skewed. Proper transformations are hereby defined as those wherein either (a) the shift along the time axis and the coordinates in E^3 are functionally independent, or (b) the functional dependence between the time coordinate and the space coordinates of E^3 is an expression of a physical theory specifically dealing with the relations between them. All other ${}^4T$ transformations are defined to be skewed.

This classification of transformations, in either E^4 or R^4, and the attendant restriction to proper transformations ${}^4T$, form the foundation of the remainder of the discussion. The importance of this distinction is that it allows easy separation of those distortions of the resultant metric which are caused by the choice of transformation types from those which are caused by phenomena in the physical world, such as the motion of matter in the universe. Obviously this separation becomes a trivial problem when ${}^3T$ transformations are used, because any distortion may be ascribed entirely to physical causes. Simplicity in the interpretation of results is, therefore, the chief virtue of ${}^3T$ transformations.

Relativistic transformations. Perhaps the best known, proper, ${}^4T$ transformation is the Lorentz translational transformation expressing the relation between a system y^i moving with a relative velocity v in the positive x^3 direction; namely,

$$(7) \qquad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = \gamma(x^3 - vx^4), \quad y^4 = \gamma(x^4 - \alpha_0 x^3),$$

in which the transformation parameters γ and α_0 are given by

$$(8) \qquad \gamma=(1-v^2c^{-2})^{-1/2}, \qquad \alpha_0=vc^{-2},$$

involving the velocity of light as measured by an observer in the x^i system, along with the previously defined transformation parameter v.

As explained in the Appendix, it is also possible to define a proper ${}^4T$ rotational transformation in R^4, given by

$$(9) \qquad Z^1=X^1, \quad Z^2=\gamma(X^2-\Omega X^4), \quad Z^3=X^3, \quad Z^4=\gamma(X^4-\alpha X^2),$$

which corresponds to an angular velocity Ω about the X^3 axis. The transformation parameters α, γ and r are defined as

$$(10) \qquad \alpha=\Omega^{-1}c^{-2}v^2, \qquad \gamma=(1-v^2c^{-2})^{-1/2}, \qquad r=|X|^1=|Z|^1,$$

involving the velocity of light, c, as measured by an observer in the X^i system, and the local velocity v, which is equal to the product of the angular velocity and the local radius. Note that r is not *identical* to either X^i or Z^i.

As in the case of equation (7), the curvature of the space is not altered. In particular,

$$K_{11}=1, \quad K_{22}=(Z^1)^2, \quad K_{33}=1, \quad K_{44}=-c^2, \qquad \text{all other } K_{ij}=0.$$

At this step this analysis departs from those of Schiff and Trocheris. Although they both displayed a non-singular relationship between the stationary and the moving reference frames, neither appeared to appreciate its significance; namely, that such a moving frame of reference must of necessity span a flat space[3]. Because of this flatness it is possible to define a rotating, orthogonal, coordinate system that is linearly related to the stationary system. One such frame is that defined by (9).

Before inquiring as to the conditions necessary for the above determination of the vector form of Maxwell's equations in a rotating system in E^3, note that the circumferential Lorentz contraction mentioned by Møller **[8]** and by Einstein **[15]** may be obtained from the inverse of (9); that is, that

$$r dX^2=\gamma r dZ^2=r dZ^2/(1-v^2c^{-2})^{-1/2}.$$

Recall that X^i has been chosen as the preferred, or stationary, system and that the X^i describe a space of zero curvature. Thus the preferred system is in a gravity-free region, and the geodesics of this space, which will be denoted by S_1, are indeed straight lines.

The events that take place in S_1 may be viewed by observers in motion with respect to S_1 as well as by observers at rest with respect to S_1. Consider, for the moment, three representative observers, named A, B, and C. Suppose

3) Trocheris observed that the rotational space was flat, but did not pursue the implications of this flatness.

A is moving along a geodesic (in this case, a straight line) with a velocity v, and that B and C are undergoing rotation with respect to the coordinates X^i, which are stationary with respect to S_1. Let y^i be the coordinates of a rectangular cartesian system moving with A, and let z^i be the coordinates rotating with B and C. That the picture of S_1 available to B and C can be correctly determined, within the scope of special relativity, by means of (9) has been assumed by Einstein [15] and confirmed by the experiments reported by Sherwin.

At his position, if A's velocity is constant, he can not only observe S_1 from his moving frame, but he can also compute the outcome of events in S_1 from variables measured in terms of y^i by means of formulas written for S_1 in terms of x^i.

Clearly B and C are not so fortunate. Suppose that B wishes to compute the outcome of events in S_1 by means of formulas written for S_1, but with x^i replaced by z^i. His efforts will often meet with failure, as predicted by the general theory of relativity. It is precisely in cases of this sort that the relative rotation of the matter of the universe must be taken into account. If B computes geodesics on the basis of this relative rotation of the matter in the universe he will certainly arrive at geodesics different from those in S_1, which geodesics now describe a new space S_2. It is this space S_2 which Schiff erroneously associated with the moving coordinates described by his transformation. That he did not arrive at S_1, or any equivalent space, is forcefully indicated by his failure to find Maxwell's equations invariant, contrary to the fundamental postulate of the general theory of relativity[4].

Profiting by the experience of B, C may decide to record the events in S_1 in terms of the z^i coordinates available to him and deduce new formulas for determining the outcome of events in S_1, but phrased in terms of z^i. In so doing, he will obtain modified forms of Coriolis forces, centrifugal forces, and so on. Moreover, if it were not for the Lorentz contraction, he would obtain the Newtonian expressions for Coriolis forces, centrifugal forces, and the like. These forces associated with rotating systems are not directly due to the motion of distant masses, but due to the fact that additional forces must be applied to a particle to cause it to move along lines which are not geodesics of S_1. Rather, their connection is indirect, in that the matter of the universe determines the nature of the geodesics of S_1.

In our determination of the form of the Maxwell's equations we shall adopt the attitude, and viewpoint, of observer C. Our desire is to abide by the physical laws that have been found to hold in a preferred system, as deter-

4) This fundamental postulate is frequently referred to as the Principle of Covariance, a phrase apparently coined by Einstein. It has occasionally been misconstrued to mean that physical law are invariant if expressed in terms of covariant tensors. See Tolman [16] and Fok [17].

mined by general relativity, but to observe the course of events in S_1 from a coordinate frame that is in rotation relative to the geodesic coordinate system of S_1. In so doing, we further wish to take cognizance of the Lorentz contraction in relating the view of event as seen from the rotating system with that of the same event as viewed by a stationary observer.

Maxwell's equations in a rotating coordinate system. In terms of the Minkowski tensor, defined in R^4, Maxwell's equations assume the form [18], [19]:

$$F_{ij,k}+F_{jk,i}+F_{ki,j}=0\,, \tag{11}$$

$$b^{-1/2}(b^{1/2}F^{ij})_{,j}=\mu_0 I^i\,, \tag{12}$$

where $\partial F_{ij}/\partial x^k \equiv F_{ij,k}$ and where $F_{ij}=-F_{ji}$, and the only non-zero components are

$$\begin{aligned} &F_{12}=B_3\,, \quad &F_{13}=-B_2\,, \quad &F_{14}=E_1\,, \\ &F_{23}=B_1\,, \quad &F_{24}=E_2\,, \quad &F_{34}=E_3\,, \end{aligned} \tag{13}$$

in which B_1, B_2, B_3 and E_1, E_2, E_3 are the components of the magnetic and electric fields respectively in the directions indicated by their subscripts. In equation (12) the determinant of the metric of the coordinate system in which (12) is to be evaluated is denoted by b, the 4-current vector in R^4 is denoted by I^i, and permeability is denoted by μ_0.

Upon studying equations (11), (12), and (13) it becomes evident that C need not adhere to his previous choice in the case of this particular set of relations. He may take advantage of the fact that the electromagnetic field vectors may be derived from a potential which is not itself affected by an observer's measurement of it from a set of rotating coordinates. Measurements found under these circumstances will not be identical with those found by a stationary observer, but they will define the field quantities as detected in a rotating system. Should the observer be skeptical of this argument, he may evaluate (11), (12) and (13) in S_1, as originally planned, and then transform his findings to the rotating system. In either case he will find that

$$\nabla'\times\bar{H}'=\frac{\partial\bar{D}'}{\partial t'}+\bar{J}'\,,\quad \nabla'\times\bar{E}'=-\frac{\partial\bar{B}'}{\partial t'}\,,\quad \nabla'\cdot\bar{D}'=\rho'\,,\quad \nabla'\cdot\bar{B}'=0\,. \tag{14}$$

All quantities and operators appearing in (14) are defined in terms of the rotating coordinate system.

All that remains is to display the relations between field quantities measured by observers undergoing relative rotation. We have that, upon using primed symbols in the moving system and unprimed in the stationary system,

$$B_1'=-x^2(x^1B_2-x^2B_1)r^{-2}+\gamma x^1(x^2B_2+x^1B_1)r^{-2}-\gamma\Omega c^{-2}x^1E_3\,, \tag{15}$$

$$B_2' = x^1(x^1B_2 - x^2B_1)r^{-2} + \gamma x^2(x^2B_2 + x^1B_1)r^{-2} - \gamma\Omega c^{-2}x^2E_3\,, \tag{16}$$

$$B_3' = \gamma B_3 + \gamma\Omega c^{-2}(x^1E_1 + x^2E_2)\,, \tag{17}$$

$$E_1' = x^2(x^2E_1 - x^1E_2)r^{-2} + \gamma x^1(x^1E_1 + x^2E_2)r^{-2} + \gamma\Omega x^1B_3\,, \tag{18}$$

$$E_2' = -x^1(x^2E_1 - x^1E_2)r^{-2} + \gamma x^2(x^1E_1 + x^2E_2)r^{-2} + \gamma\Omega x^2B_3\,, \tag{19}$$

$$E_3' = \gamma E_3 - \gamma\Omega(x^2B_2 + x^1B_1)\,, \tag{20}$$

$$J^{1\prime} = x^1(x^1J^1 + x^2J^2)r^{-2} + \gamma x^2(x^2J^1 - x^1J^2)r^{-2} + \gamma\Omega x^2\rho\,, \tag{21}$$

$$J^{2\prime} = x^2(x^1J^1 + x^2J^2)r^{-2} + \gamma x^1(x^1J^2 - x^2J^1)r^{-2} - \gamma\Omega x^1\rho\,, \tag{22}$$

$$J^{3\prime} = J^3\,, \tag{23}$$

$$\rho' = \Omega c^{-2}(x^2J^1 - x^1J^2) + \gamma\rho\,. \tag{24}$$

For the sake of completeness, the vector relations are

$$\bar{E}' = (1-\gamma)(\hat{v}\cdot\bar{E})\hat{v} + \gamma\left[\bar{E} + (\bar{v}\times\bar{B})\right], \tag{25}$$

$$\bar{B}' = (1-\gamma)(\hat{v}\cdot\bar{B})\hat{v} + \gamma\left[\bar{B} - \frac{1}{c^2}(\bar{v}\times\bar{E})\right], \tag{26}$$

$$\bar{J}' = \bar{J} - \gamma\rho\bar{v} - (1-\gamma)(\hat{v}\cdot\bar{J})\hat{v}\,, \tag{27}$$

$$\rho' = \gamma\left[\rho - \frac{1}{c^2}(\bar{\Omega}\times\bar{r})\cdot\bar{J}\right], \tag{28}$$

wherein $\bar{r} = (\hat{\Omega}\times\bar{R})\times\hat{\Omega}$, with $\bar{R} = \hat{i}X^1 + \hat{j}X^2 + \hat{k}X^3$ [5].

Example. Suppose that in a stationary frame of reference an electric field in the vertical direction is imposed upon a circular current loop of radius a in the horizontal plane. If we assume zero total charge, the only forces acting upon the loop will be in the plane of the loop. Hence

$$\rho = 0\,, \qquad \bar{J} = (\bar{\Omega}\times\bar{a})\rho_-\,, \qquad \bar{F}_e = \int_\eta \rho\bar{E}d\eta\,,$$

$$\bar{v} = \bar{\Omega}\times\bar{a}\,, \qquad \bar{\Omega} = \hat{k}\Omega\,, \qquad \bar{F}_m = \int_\eta \bar{J}\times\bar{B}d\eta\,,$$

which is to say that the current is composed of charge ρ_- revolving with a uniform angular velocity Ω, and that the integral over all space, η, of $(\rho_+ + \rho_-)\bar{E} = \rho\bar{E}$ is zero because $\rho = 0$.

An observer in a coordinate system subjected to an angular velocity Ω about a vertical axis concentric with the current loop will find, according to equations (25) through (28), that

5) Likewise, it is possible to show that $\nabla' = \nabla - (1-\gamma)\hat{v}(\hat{v}\cdot\nabla) + \alpha\gamma\hat{v}\frac{\partial}{\partial t}$, and $\frac{\partial}{\partial t'} = \gamma\left[\frac{\partial}{\partial t} + (\Omega\times\bar{r})\cdot\nabla\right]$.

$$\bar{E}' = \gamma\left[\bar{E} + \Omega a(\hat{k} \times \hat{r}) \times \bar{B}\right], \tag{29}$$

$$\bar{B}' = \gamma\left[\bar{B} - \frac{\Omega a}{c^2} E\hat{r}\right], \tag{30}$$

$$\bar{J}' = \gamma \Omega a \rho_{-}(\hat{k} \times \hat{r}), \tag{31}$$

$$\rho' = -\frac{\Omega^2 a^2}{c^2}\gamma \rho_{-}, \tag{32}$$

where $\hat{k}$ and $\hat{r}$ are unit vectors in the vertical and radial directions respectively.

The total force acting on the loop is then

$$\bar{F}'_e + \bar{F}'_m = \int_\eta [\rho' \bar{E}' + \bar{J}' \times \bar{B}'] d\eta, \tag{33}$$

in which the integrand may be written as

$$\begin{aligned} \rho'\bar{E}' + \bar{J}' \times \bar{B}' &= -\gamma^2 \frac{\Omega^2 a^2}{c^2}\rho_{-}\left[\bar{E} + \Omega a(\hat{v} \times \bar{B})\right] + \gamma^2 \Omega a \rho_{-} \hat{v} \times \left[\bar{B} - \frac{\Omega a}{c^2} E\hat{r}\right] \\ &= -\gamma^2 \frac{\Omega^2 a^2}{c^2}\rho_{-} E\left[\hat{k} + (\hat{v} \times \hat{r})\right] + \gamma^2 \Omega a \rho_{-}\left(1 - \frac{\Omega^2 a^2}{c^2}\right)(\hat{v} \times \bar{B}) = \Omega a \rho_{-}(\hat{v} \times \bar{B}), \end{aligned}$$

which indicates that the moving observer will find a force identical to that found by the stationary observer.

It is interesting to notice that the so-called "second-order" terms play an important part in establishing the result just obtained. Consequently the conservation of charge, often justified by first-order approximations, must be considered in R^4 in terms of the scalar invariant

$$I^i I_i = \Phi^2. \tag{34}$$

Thus the principle of conservation of charge becomes

$$\Phi = \text{constant},$$

which includes the statement of the principle as given by Landau and Lifshitz [20]. In the example of the ring current,

$$\bar{J} \cdot \bar{J} - c^2 \rho^2 = \bar{J}' \cdot \bar{J}' - c^2 \rho'^2,$$

which may be verified by substituting into (34) according to equations (31) and (32), and the values given in the statement of the problem. Thus

$$\begin{aligned} \Omega^2 a^2 \rho_{-}^2 &= \gamma^2 \Omega^2 a^2 \rho_{-}^2 - \gamma^2 \frac{\Omega^4 a^4}{c^2}\rho_{-}^2 \\ &= \gamma^2 \Omega^2 a^2 \rho_{-}^2 \left(1 - \frac{\Omega^2 a^2}{c^2}\right) = \Omega^2 a^2 \rho_{-}^2, \end{aligned}$$

which demonstrates that equation (34) is satisfied.

Concluding remarks. To concatenate the highlights of the previous analysis, the problem of determining the vector form of Maxwell's equations in a rotating frame in E^3 by recourse to their invariant form in R^4 may be recast in a somewhat simpler form than presented thus far. First of all, this is a problem in special relativity because it is concerned with the relative motion of two coordinate systems which are dependent according to an explicit, non-singular, relation[6]. Secondly, the curvature of space as measured in the moving frame will be the same as that measured in the stationary frame of reference because the tensor equations

$$R^{i}_{\cdot jkl}=0 \tag{35}$$

for the components of the Riemann curvature tensor hold for both systems, by virtue of their explicit, non-singular, dependence. Thirdly, equations (35) assure the existence of a rotating rectangular Cartesian coordinate system which spans the space. And last, since the metric of the rotating rectangular Cartesian frame is of the same form as the metric of the stationary rectangular Cartesian frame, it follows from (11), (12), and (13) that equation (14) holds in the moving system.

Although equations (15) through (24) are implied by (14), they cannot be written down without reference to the explicit transformation relating the two reference frames in relative rotation. Inasmuch as the form of this transformation in the E^3 hypersurface in R^4 is well known and accepted, the only outstanding question is that of the form of the transformation in R^4. Relation (2) is rejected as being unwieldy, in terms of conclusions drawn from (35), since only Z^2 is subjected to a pseudo-rotation. Since (5) represents a non-relativistic, or Newtonian, rotation, it is rejected in favor of (9), which incorporates the circumferential and temporal contractions consistent with the special theory of relativity. Although the justification of the last line of (9) is more difficult than in the case of pure translation, it follows directly from the first three, whose physical interpretation is obvious. Details and further consideration of possible transformations will be deferred to the Appendix.

Instead of direct relationships between (14), evaluated in a stationary system, and (14′), say, evaluated in a moving system, relationships of the following nature exist: Given a set E_i and B_i $(i=1,2,3)$, satisfying equations (14) in a stationary system, there exists a set E_i' and B_i' $(i=1,2,3)$ related to E_i and B_i according to equations (15) through (20), such that each member of the set will satisfy a set of equations (14′) which are related to (14) according to equations (21) through (28).

6) One of the major problems of the general theory is the determination of the metric of a curved space. This is because no such single-valued, one-to-one relation can be determined. See, for example, Rainich, Chap. 5, or Synge, Chaps. 7 and 8.

To demonstrate, suppose

$$B'_{3,2}-B'_{2,3}-c^{-2}E'_{1,4}=\mu_0 J^{1\prime} ,$$

where B'_i, E'_i, $J^{i\prime}$ and ρ' are given by (15) through (24) in terms of B_i, E_i, J^i, and ρ, which satisfy (14). Upon evaluating the left-hand side of (29) we find that

$$(B_{3,2}-B_{2,3}-c^{-2}E_{1,4})\left[(x^1)^2+\gamma(x^2)^2\right]r^{-2}+(B_{1,3}-B_{3,1}-c^{-2}E_{2,4})(1-\gamma)x^1x^2r^{-2}$$
$$+(E_{1,1}+E_{2,2}+E_{3,3})c^{-2}\gamma\Omega x^2=\mu_0 J^{1\prime} .$$

Recalling (14), we may write

$$\begin{aligned}&\mu_0 J^1\left[(x^1)^2+\gamma(x^2)^2\right]r^{-2}+\mu_0 J^2(1-\gamma)x^1x^2r^{-2}+\mu_0\rho\gamma x^2\Omega\\&=\mu_0\left[x^1(x^1J^1+x^2J^2)r^{-2}+\gamma x^2(x^2J^1-x^1J^2)r^{-2}+\gamma\Omega\rho x^2\right]\\&=\mu_0 J^{1\prime}\end{aligned}$$

in agreement with (21). Similar equalities may be obtained from the remaining equations in (14′).

Both Schiff and Trocheris considered the following paradox attributed by Schiff to Oppenheimer:

"Consider two concentric spheres with equal and opposite total charges uniformly distributed over their surfaces. When the spheres are at rest, the electric and magnetic fields outside the spheres vanish. When the spheres are in uniform rotation about an axis through their center, the electric field outside vanishes, while the magnetic field does not, since the magnetic moment of each of the spheres is proportional to the square of its radius. Suppose that the spheres are stationary; then an observer traveling in a circular orbit around the spheres should find no field, for since all of the components of the electromagnetic field tensor vanish in one coordinate system, they must vanish in all coordinate systems. On the other hand, the spheres are rotating with respect to this observer, and so he should experience a magnetic field."

Of course, the simplest solution to the paradox may be found in the observation that the paradox exists only if one fails to observe that the tensor argument used in the formulation of the paradox applies only to equations (11), (12), and (13), but not to equation (14). Hence, the paradox disappears when a proper transformation is used.

Schiff attempted a solution by neglecting second order and higher terms in his result and by employing a first order perturbation calculation.

In place of relations similar to those of the stationary system, the moving observer finds that

$$(E_{1,1}+E_{2,2}+E_{3,3})c^{-2}\gamma\Omega x^2=\mu_0 J^{1\prime}=\gamma\Omega x^2\mu_0\rho ,$$

indicating that the fields he measures are consistent with the currents and charges that can be detected. Likewise, an argument similar to that used by Webster may be used in conjunction with the results obtained herein to explain unipolar induction.

Invariance of Maxwell's equations to rotation is a satisfying companion to their invariance to translation. As pointed out by Trocheris, it certainly seems reasonable that an observer moving with a velocity $v=\Omega r$ along the circumference of a very large circle and an observer moving with a velocity v along a tangent to the circle should arrive at similar electromagnetic observations at their very instant of tangency.

To be more precise, let $v=\Omega r$ as r increases to infinity. Without any loss of generality (because of cylindrical symmetry), x^2 may be set equal to zero, so that $x^1=r$, implying that the instantaneous velocity is in the positive x^2 direction. Equations (15) through (24) then become

$$\begin{aligned}
&\rho' = \gamma(\rho-\Omega r c^{-2} J^2)\,, \\
&J^{1\prime} = J^1\,, \qquad J^{2\prime} = \gamma(J^2 - v\rho)\,, \qquad J^{3\prime} = J^3\,, \\
&H_1' = \gamma(H_1 - vc^{-2}E_3)\,, \quad H_2' = H_2\,, \qquad H_3' = \gamma(H_3 - vc^{-2}E_1)\,, \\
&E_1' = \gamma(E_1 + vH_3)\,, \qquad E_2' = E_2\,, \qquad E_3' = \gamma(E_3 + vH_2)\,,
\end{aligned}$$

so that the field quantities, charges, and current densities found in the rotating system indeed approach those found in the translating system in the limit of increasing radius r **[21]**.

If Schiff's results are rewritten in terms of an orthogonal reference system they become identical with those obtained through the use of (5), which is a Newtonian rotation extended to R^4. Inasmuch as Trocheris considered only motion in which $\gamma\simeq 1$, the essential difference between his results and those of Schiff lies in the choice of a different non-orthogonal coordinate system, which accordingly demands the inclusion of θ-dependent terms. As before, these expressions may be rewritten in terms of an orthogonal reference system. In that form the θ-dependence no longer appears, and formal agreement with Schiff's results is reached.

All three results thus require no direct appeal to the general theory of relativity. Whether described by the present orthogonal, or the earlier non-orthogonal, reference frame, a direct calculation shows that all components of the Riemann curvature tensor are zero in the rotating coordinate system. Thus it is evident that the off-diagonal terms found by previous investigators indicate nothing more than a mathematically engendered non-orthogonality. Rotational invariance of Maxwell's equations can, as noted earlier, be proven on this basis alone.

Finally, it is possible to exhibit transformations between fixed coordinate systems in E^3 involving transformation parameters similar to those in equation

(9). Consider, for example,

$$X^1 = X'^1 , \quad X^2 = rX'^2 , \quad X^3 = X'^3 , \quad r \equiv |X^1| ,$$

in which the primed system differs from the unprimed only in that the angular units vary inversely with radial distance. Since the geometry is independent of any choice of units, it follows that r plays the role of a constant in the evaluation of the $\partial X^m / \partial X'^i$ appearing in

$$G'_{ij} = G_{mn} \frac{\partial X^m}{\partial X'^i} \frac{\partial X^n}{\partial X'^j} .$$

Appendix. In what follows, the steps leading to (9) will be reviewed first, and then a thought experiment, of the sort introduced by Einstein, appropriate to a rotating frame of reference will be outlined. This experiment is not essential to the foregoing development—it is presented only for those who wish to associate (9) with such an experiment.

Underlying the justification of (9) is the observation that if two sets of coordinates, X^i and Z^i, are each sufficient to describe all points of space-time, then the Riemann curvature tensor $\tilde{R}^i_{.jkl}$ in the rotating system is related to $\tilde{R}^i_{.jkl}$ in the stationary system through

$$\tilde{R}^i_{.jkl} = R^m_{.npq} Z^i_m X^n_j X^p_k X^q_l ,$$

since $X^i = X^i(Z^k)$ and $Z^i(X^k)$, $i, k = 1, 2, 3, 4$, are defined throughout the space such that

$$X^m_i \equiv \frac{\partial X^m}{\partial Z^i} \quad \text{and} \quad Z^m_i \equiv \frac{\partial Z^m}{\partial X^i}$$

exist and are continuous everywhere. Thus $\tilde{R}^i_{.jkl}$ is zero if $R^i_{.jkl}$ is zero, and (35) therefore applies to both coordinate systems. Moreover, condition (35) is sufficient to assure the existence of an orthogonal coordinate system in each case.

Coordinates X^i have been chosen to be orthogonal, consequently it only remains to exhibit an orthogonal set of rotating coordinates Z^i. In E^3 such a set of rotating cylindrical coordinates must be related to a similar stationary set by a form equivalent to the first three lines of (36).

$$(36) \qquad Z^1 = X^1 , \; Z^2 = \gamma(X^2 + \Omega X^4) + \varphi , \; Z^3 = X^3 , \; Z^4 = aX^2 + bX^4 .$$

Thus the problem of specifying Z^i may be reduced to the problem of specifying Z^4.

By an appropriate choice of transformation parameters (36) can be made to correspond to (2), (5), or (9). Since (2) has already been discarded in favor of (5), and since (5) corresponds to a Newtonian transformation, the existence of an orthogonal system Z^i essentially different from either (2) or (5) depends upon the existence of an essentially different set of transformation parameters a, b, γ, and φ.

Having discarded (2), parameter φ may be set equal to zero by choosing Z^2 equal to zero when X^2 and X^4 are both equal to zero. The nature of the remaining parameters may be determined from the required orthogonality of Z^i. The determination may be shortened somewhat if a and b are replaced by

$$a = -\alpha\beta\,, \quad b = \beta\,, \tag{37}$$

where the earlier definitions of α and γ are suspended. Thus

$$\begin{aligned} &Z^1 = X^1\,, \quad Z^2 = \gamma(X^2 - \Omega X^4)\,, \quad Z^3 = X^3\,, \quad Z^4 = \beta(-\alpha X^2 + X^4)\,, \\ &X^1 = Z^1\,, \quad X^2 = \Delta(\beta Z^2 + \gamma\Omega Z^4)\,, \quad X^3 = Z^3\,, \quad X^4 = \Delta(\alpha\beta Z^2 + \gamma Z^4)\,, \\ &\Delta^{-1} = \gamma\beta(1-\alpha\Omega)\,. \end{aligned} \tag{38}$$

If, in the rotating frame, $(ds)^2 = K_{ij}dZ^i dZ^j$, then

$$K_{ij} = G_{mn}X_i^m X_j^n\,, \tag{39}$$

where $(ds)^2 = G_{ij}dX^i dX^j$. From the known components of G_{ij}, i.e.,

$$G_{11} = G_{33} = 1\,, \quad G_{22} = (X^1)^2 = r^2\,, \quad G_{44} = -c^2\,,$$

the component of K_{ij} may be computed from (39) once the necessary derivatives are found from (38). Thus

$$K_{11} = K_{33} = 1\,, \; K_{22} = \Delta^2\beta^2(r^2 - c^2\alpha^2)\,, \; K_{24} = \Delta^2\gamma\beta(r^2\Omega - c^2\alpha)\,, \; K_{44} = \Delta^2\gamma^2(r^2\Omega^2 - c^2)\,.$$

Orthogonality will hold only if $K_{24}=0$. Now if the transformation is to be non-singular, it is necessary that $\Delta \neq 0$ and $\gamma\beta \neq 0$. Consequently $K_{24}=0$ implies that

$$\alpha = r^2\Omega c^{-2} = \Omega^{-1}v^2c^{-2}\,. \tag{40}$$

According to (36), the Z^1 and Z^3 axes are mutually orthogonal because they are parallel to X^1 and X^3 respectfully. Therefore it is necessary to ascertain the Z^4 direction perpendicular to Z^2 and perpendicular to the X^1X^3, or Z^1Z^3, plane, which is unaltered by (36). Since the choice of the unit of length in the Z^4 direction does not affect its direction, no generality is lost by setting $G_{44} = K_{44} = -c^2$, which yields that

$$-\Delta^2\gamma^2 = (1 - v^2c^{-2})^{-1}\,. \tag{41}$$

Again with no loss of generality, K_{22} may be taken equal to r^2, so that

$$\Delta^2\beta^2 = (1 - v^2c^{-2})^{-1}\,. \tag{42}$$

Upon dividing (41) by (42) it is evident that

$$\gamma^2 = \pm\beta^2,$$

wherein the choice of algebraic sign depends upon the choice relative rotation of the two systems. Adoption of the positive sign leads to

$$\Delta^{-1} = 1\,, \quad \gamma^2 = (1-v^2c^{-2})^{-1}\,. \tag{43}$$

If initially γ had been set equal to unity, this would have resulted in $\beta=1$ and $\Omega=0$, implying that only (5) is suitable under such a restriction.

Recall that a, b, and γ were defined as transformation parameters in (36), so that according to (37) α, β, and γ are also transformation parameters. Therefore they play the role of constants in the evaluation of X^m_i and Z^i_m. Only in this way can the orthogonality of the Z^i system, consistent with (35), be maintained. Finally, the substitution of the representations found into (36) shows it to be equivalent to (9).

These conclusions may be reaffirmed by assuming that α and γ are not to be treated as transformations parameters. Consider the expressions

$$X^1 = Z^1\,, \quad X^2 = \gamma(Z^2+\Omega Z^4)\,, \quad X^3 = Z^3\,, \quad X^4 = \gamma(Z^4+\alpha Z^2)\,, \tag{44}$$

obtained by solving (9) for X^1 through X^4. Their derivatives then become

$$\begin{aligned} &X^1_1 = X^3_3 = 1\,, \quad X^2_1 = \frac{\Omega^2\gamma^2}{c^2}X^1X^2\,, \quad X^2_2 = \gamma\,, \quad X^2_4 = \gamma\Omega\,, \\ &X^4_1 = \frac{\Omega\gamma}{c^2}X^1(\Omega\gamma X^4+2Z^2)\,, \quad X^4_2 = \alpha\gamma\,, \quad X^4_4 = \gamma\,. \end{aligned} \tag{45}$$

It then follows from (39) that

$$\begin{aligned} &K_{11} = K_{33} = 1\,, \\ &K_{21} = G_{22}X^2_2X^2_1 + G_{44}X^4_2X^4_1 = K_{12}\,, \\ &K_{22} = G_{22}X^2_2X^2_2 + G_{44}X^4_2X^4_2\,, \\ &K_{24} = G_{22}X^2_2X^2_4 + G_{44}X^4_2X^4_4 = K_{42}\,, \\ &K_{41} = G_{22}X^2_4X^2_1 + G_{44}X^4_4X^4_1 = K_{14}\,, \\ &K_{44} = G_{22}X^2_4X^2_4 + G_{44}X^4_4X^4_4\,, \\ &\text{all other } K_{ij}=0\,. \end{aligned} \tag{46}$$

The Z^i system will be orthogonal if

$$K_{21} = K_{24} = K_{41} = 0\,, \tag{47}$$

everywhere. Equations (45), (46), and (47) together are satisfied only if

$$r^2 = c^2\Omega^{-2} = (X^1)^2 \quad \text{and} \quad \gamma(X^2-\Omega X^4) = 2Z^2 = Z^2\,.$$

But these relations hold only at $Z^2=0$ and $Z^1=$constant, that is, on a hypersurface of R^4, so that Z^i is not orthogonal everywhere.

There is yet the possibility of redefining α, β, and γ such that (36), (37), (47) may all be satisfied. If the last line of (44) is replaced by the last line of (38), then

$$X^2_1 = (\Delta\beta)_{,1}Z^2 + (\Delta\gamma)_{,1}\Omega Z^4 , \quad X^4_1 = (\Delta\alpha\beta)_{,1}Z^2 + (\Delta\gamma)_{,1}Z^4 ,$$
$$X^2_2 = \Delta\beta , \quad X^2_4 = \Delta\gamma\Omega , \quad X^4_2 = \Delta\alpha\beta , \quad X^4_4 = \gamma\Delta ,$$

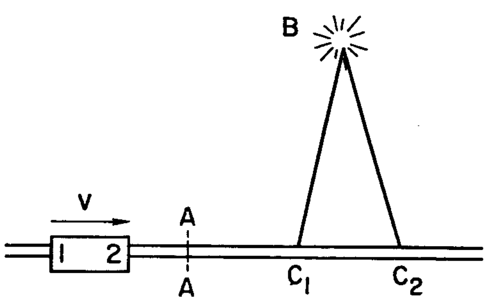
where differentiation is indicated by a comma. In this case (47) may be satisfied if $\alpha = \Omega^{-1}$. However $K_{24} = 0$ itself further demands that

$$(X^1)^2 = c^2\Omega^{-2} . \tag{48}$$

Since (48) restricts (47) to the hypersurface X^1=constant, this alone shows that there is no possible choice of α, β, and γ which meets conditions (35), (38), and (47) throughout R^4, unless α, β, and γ are recognized as transformation parameters.

Equation (9) may also be justified on physical grounds by following a line of reasoning similar to that used by Einstein [3] in his determination of the Lorentz transformation for linear motion. In both linear and rotary motion the reasoning is based upon the constancy of the velocity of light in any direction, including the direction of motion, regardless of the motion of the reference system. Time and length in the direction of motion may, therefore, be related in terms of the invariant velocity of light. Before describing the experiment in rotating coordinates it is necessary to examine the translatory experiment sufficiently carefully to appreciate that the nature of the Lorentz contraction is entirely due to the fact that time and length in the direction of motion are related exclusively by means of the invariant velocity of light in a direction parallel to the motion. Although the setting of clocks is necessary in the translational experiment, it is the *manner* in which clocks are set that is important[7]; the setting of clocks in the rotating system by means of a light pulse from the center is of no value here because the light does not travel

7) If synchronization of clocks were or primary concern in the case of translation, it could be achieved in the following manner: Knowing the length and velocity of the measuring rod, as seen by the stationary observer, it is possible to position A and B so that a flash may be emitted from B, at a certain time after the observers have passed A, such that observer 1, now at C_1, will see the flash at the same instant that it is seen by observer 2, now at C_2. Although the moving observers at C_1 and C_2 have now synchronized their clocks, their achievement is of no help in relating time and length in terms of relative velocity and the constant velocity of light.



parallel to the direction of motion.

Bearing this in mind, measurements similar to the translational ones may be performed by an observer on a rotating platform of the shape suggested by Phipps; namely, by an n-shaped polygon such that the length of each side is $2R\sin\frac{\pi}{n}$, where R is the distance from the center to the vertex of the angle formed by two adjacent sides of the polygon. The axis of rotation is normal to the plane of the platform and through the center of the polygon. By mounting a plane mirror at each vertex, perpendicular to the radius, a ray of light emitted parallel to one side of the polygon will propagate around, and parallel to, the periphery of the polygon. As $n\rightarrow\infty$ the polygon will approach a circle, and the nature of the propagation will become independent of the angular velocity. If the polygon and the limiting process are duplicated in the stationary system, similar measurements may be compared in a manner parallel to that used in the case of translation. The results may be given by (9).

Clearly one cannot replace X^2 in the last line of (9) by X'^2 where

$$X'^2 = X^2 - 2n\pi, \qquad n = \max\{m|2\pi m < X^2\}, \qquad m = 0, 1, 2, \cdots,$$

because this is to ignore the rotation through $2n\pi$ radians which is still included in the second line of (9). A similar change in both expressions is equivalent to a shift of the special reference. Thus (9) is a single-valued function of X^2.

Once the Lorentz transformation (9) for rotation has been formulated, all clocks on the rotating platform may be re-synchronized upon a signal from the center, and their timekeeping observed as the rotation continues. Suppose two clocks, at equal distance from the center, are at $X^2_{(0)}(P)$ and $X^2_{(0)}(Q)$ at the moment when all clocks are set and when the coordinate systems are coincident, that is, when $X^2=0$ at $X^4=0$. Then

$$Z^2(P) = \gamma[X^2(P) - \Omega X^4], \qquad Z^2(Q) = \gamma[X^2(Q) - \Omega X^4]$$

and

$$\left.\begin{aligned} Z^4(P) &= \gamma[X^4 - \alpha X^2(P) + \alpha X^2_{(0)}(P)], \\ Z^4(Q) &= \gamma[X^4 - \alpha X^2(Q) + \alpha X^2_{(0)}(Q)]. \end{aligned}\right. \qquad 0 \leq X^2_{(0)} < 2\pi,$$

With P and Q on a rigid disk,

$$X^2(P) - X^2_{(0)}(P) = X^2(Q) - X^2_{(0)}(Q) = \Omega X^4$$

and thus[8)]

$$Z^4(P) = Z^4(Q) = \gamma^{-1}X^4.$$

8) See Møller, p. 225.

Hence only clocks at the same radius maintain synchronization with one another.

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REFERENCES

[1] L. I. Schiff: A question in general relativity, *Proc. Nat. Acad. Sci.* (*U. S.*), **25** (1939), 391-395.

[2] M. G. Trocheris: Electrodynamics in a rotating frame of reference, *Phil. Mag.*, (7) **40** (1949), 1143-1154.

[3] A. Einstein: Zur Elektrodynamik bewegter Körper, *Ann. Phys.*, **17** (1905), 891-921.

[4] C. W. Sherwin: Some recent experimental tests on the 'Clock Paradox', *Phys. Rev.*, (2) **120** (1960), 17-21.

[5] H. Thirring: Über die Wirkung rotierender ferner Massen in der Einsteinschen Gravitationstheorie, *Phys. Zeits.*, **19** (1918), 33-39 and **21** (1921), 29-30.

[6] C. W. Berenda: The problem of the rotating disk, *Phys. Rev.*, (2) **62** (1942), 280-290.

[7] E. L. Hill: A note on the relativistic problem of uniform rotation, *Phys. Rev.*, (2) **69** (1946), 488-491.

[8] C. Møller: The theory of relativity, *Clarendon Press, Oxford*, (1952).

[9] N. Rosen: Notes on rotation and rigid bodies in relativity theory, *Phys. Rev.*, (2) **71** (1917), 54-58.

[10] A. P. Riabushko and I. Z. Fisher: The motions of rotating masses in the general theory of relativity, *Sov. Phys. JETP*, **34** (7) (1958), 822-825.

[11] H. Hönl and H. Dehnen: Zur Frage der Relativität der Rotationsbewegungen, *Zeit. Phys.*, **166** (1962), 544-558.

[12] T. E. Phipps: Re-entrant motion in special relativity, *Nature*, **195** (1962), 67-68.

[13] D. L. Webster: Schiff's charges and currents in rotating matter, *Amer. J. Phys.*, **31** (1963), 590-597.

[14] L. P. Eisenhart: Riemannian Geometry, *Princeton Univ. Press, Princeton, N. J.*, (1949).

[15] A. Einstein: Die Gründlagen der allegemeinen Relativitatstheorie, *Ann. Phys.*, **49** (1916), 769-822.

[16] R. C. Tolman: Relativity thermodynamics and cosmology, *Clarendon Press, Oxford*, (1958).

[17] V. A. Fok: The theory of space time and gravitation, *Pergamon Press, New York*, (1959).

[18] G. Y. Rainich: Mathematics of relativity, *John Wiley & Sons, Inc., New York*, (1950).

[19] J. L. Synge: Relativity: The general theory, *North-Holland Publ. Co., Amsterdam*, (1960).

[20] L. Landau and E. Liftshitz: The classical theory of fields, *Addison-Wesley Press, Inc., Cambridge, Mass.*, (1951).

[21] J. A. Stratton: Electromagnetic theory, *McGraw-Hill Book Co. Inc., New York*, (1941).